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POLYNOMIAL INTERPOLATION OF REAL FUNCTIONS I: INTERPOLATION IN AN INTERVAL

by

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Polynomial Interpolation of Real Functions I: Interpolation in an Interval

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Abstract

This paper is the first in a series to analyse the accuracy of polynomial interpolation of functions and its dependence on the locations of the interpolation nodes. It surveys known results for polynomial interpolation in an interval. It also introduces the concept of the Minimal Interpolation Sets which are the "optimal" interpolation sets. New results concerning the properties of the minimal sets as well as procedures for locating the minimal sets are presented. The table for the minimal sets in the L^∞ norm is given. An adaptive scheme for determining the interpolation order is also presented. Examples show the efficacy of this approach.

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I. Introduction

Problem of polynomial interpolation of functions is a classical topic in approximation theory. It is well known that the positions of the interpolation nodes influence drastically the quality of the approximation for high degree polynomials. Various theorems addressing this issue are known for interpolation in an one dimensional interval, particularly through the work of P.L. Chebyshev. In contrast, in two and three dimensions when the domain of interpolation is not a rectangle or a cube, little is known concerning the approximation accuracy of the interpolation.

Accurate polynomial approximation of functions is critical for the success of Finite Element computation, especially for the p and h-p versions of the Finite Element method. Functions as well as shape informations can thus be specified from their values at certain discrete points which can be selected *a priori*.

The present paper is the first in a series (see also [2]) to address the question of the dependence of the approximation accuracy on the interpolation nodes. It gives a survey of known results related to polynomial interpolation in an interval. It also introduces the concept of minimal interpolation sets and presents the principle behind the computational procedure used in finding these minimal interpolation sets for an interval as well as for a triangle. Finding the minimal interpolation sets for the triangle is the focus of the second paper.

In section II, we review known results pertinent to our study. Section III introduces the concept of the minimal interpolation sets for various norms. Interpolation using the minimal sets leads to the minimal interpolation error. The L^∞ minimal

sets are listed in Table 1 for polynomials up to order 20. Fig. 4 gives the graph which characterizes the error bound for such interpolation. Section IV discusses interpolation error in the derivative of the function. In Section V, we show various numerical interpolation examples. Section VI presents a procedure for the adaptive selection of the degree of the interpolation polynomial for a given error tolerance.

II. Overview of Known Results

Let $I = [-1, 1]$, $C(I)$ be the space of real continuous functions on I, $\mathcal{P}_n$ be the space of algebraic polynomials of degree less or equal to n.

Polynomial interpolation of a function is to find an approximate polynomial which takes the value of the function at some given points in I. Such polynomial always exists.

Theorem 1 Let $f(\cdot) \in C(I)$, $T = (\tau_0, \tau_1, ..., \tau_n)$, where $-1 \leq \tau_0 < \tau_1 < ... < \tau_n \leq 1$, then there is a unique polynomial $p_n(\cdot)$ in $\mathcal{P}_n$ such that $p_n(\tau_i) = f(\tau_i)$, $i = 0, 1, ..., n$. $p_n(\cdot)$ is given by Lagrange's formula

$p_n(t) = \mathcal{L}_T f(t) := \sum_{k=0}^{n} f(\tau_k) L_k(t)$, where $L_k(t) = \prod_{j=0, j\neq k}^{n} (\frac{t-\tau_j}{\tau_k - \tau_j})$.

$\mathcal{L}_T$ is a linear projection operator which maps a real continuous function to its corresponding polynomial interpolant. n is called the order of the interpolation T. τ_i are called nodes of the interpolation T.

Theorem 2 (Weierstrass) [7] Every continuous function in $C(I)$ can be uniformly approximated by polynomials to any degree of accuracy.

The question arises as what is the best polynomial approximant in $\mathcal{P}_n$ to a given function. We define the least deviation $d_n(f) := inf\{sup_{t\in I}|q(t) - f(t)|, q(\cdot) \in \mathcal{P}_n\}$.

This least deviation is always attained.

Theorem 3 [6] Given $f(\cdot) \in C(I)$, there exists $w_n(\cdot) \in \mathcal{P}_n$, such that for all $q_n(\cdot) \in \mathcal{P}_n$, $\|f - w_n\|_\infty \leq \|f - q_n\|_\infty$, where $\|\cdot\|_\infty$ is the L^∞ norm in $C(I)$, *i.e.*, $\|f\|_\infty = sup_{t\in I}|f(t)|$.

Let $\omega(\delta, f(\cdot))$ be the modulus of continuity of $f(t)$ in I, *i.e.*, $\omega(\delta, f(\cdot)) := sup_{|t_1-t_2|<\delta, t_1, t_2 \in I}|f(t_1) - f(t_2)|$, then

Theorem 4 (Jackson) [7] Let $f(\cdot)$ be an r times continuous differentiable function, f^r be its r(th) derivative, then $d_n(f) \leq A^{r+1}n^{-r}\omega(n^{-1}, f^{(r)}(\cdot))$, where $A < \pi\sqrt{3}/2$.

Therefore, the least deviation $d_n(f)$ decreases with n at least algebraically to zero with power r if f has r continuous derivative. Moreover, Bernstein showed that

Theorem 5 (Bernstein) [7] $f(\cdot)$ is analytic on I if and only if $\overline{lim_{n\to\infty}}(d_n(f))^{1/n} < 1$. $f(\cdot)$ is entire if and only if $\overline{lim_{n\to\infty}}(d_n(f))^{1/n} = 0$.

Since $\mathcal{L}_T$ is a linear projection operator from $C(I)$ to $\mathcal{P}_n$, we define the L^p norm of $\mathcal{L}_T$, $1 \leq p \leq \infty$, as $\|\mathcal{L}_T\|_p := sup_{f\neq 0, f\in C(I)}\frac{\|\mathcal{L}_T f\|_p}{\|f\|_\infty}$, where the L^p norm of a function $f \in I$ is $\|f\|_p = (\int_{-1}^1 |f(t)|^p dt)^{1/p}$.

From Theorem 1, we have

$$\|\mathcal{L}_T\|_\infty = \lambda(T) := sup_{t\in I}\sum_{k=0}^{n}|L_k(t)|.$$

The L^∞ norm $\lambda(T)$ is also called the Lebesgue constant of $\mathcal{L}_T$.

The "mean norm" of $\mathcal{L}_T$ is defined as

$$\|\langle\mathcal{L}_T\rangle\| := (\int_{-1}^1 \sum_{k=0}^{n}|L_k(t)|^2 dt)^{1/2}.$$

Note that

$$\|\mathcal{L}_T\|_p \leq |\mathcal{L}_T|_p := (\int_{-1}^{1} (\sum_{k=0}^{n} |L_k(t)|)^p dt)^{1/p}.$$

We shall call $|\mathcal{L}_T|_p$ the L^p pseudonorm or the $|L|^p$ norm of the interpolation operator $\mathcal{L}_T$.

Since $\mathcal{L}_T$ is a projection operator, $\|\mathcal{L}_T\|_p \geq 2^{1/p}$.

We emphasize that the L^p norm of $\mathcal{L}_T$ depends only on the distribution of the interpolation nodes τ_i.

How well can $\mathcal{L}_T f$ approximate f? We have the following Lebesgue inequality.

Theorem 6 For all $f(\cdot) \in C(I)$, $1 \leq p \leq \infty$, $\|f - \mathcal{L}_T f\|_p \leq (2^{1/p} + \|\mathcal{L}_T\|_p) d_n(f)$.

The proof uses the method of "intermediate approximation". Note that the n-th order interpolant of the best approximating polynomial w_n in Theorem 3 is itself, hence

$\|f - \mathcal{L}_T f\|_p = \|f - w_n + \mathcal{L}_T(w_n - f)\|_p \leq \|f - w_n\|_p + \|\mathcal{L}_T(w_n - f)\|_p \leq \|f - w_n\|_\infty (2^{1/p} + \|\mathcal{L}_T\|_p) = (2^{1/p} + \|\mathcal{L}_T\|_p) d_n(f)$.

Therefore, the interpolation error of a function is always less than a numerical factor, which does not depend on the interpolation function but depends sensitively on the distribution of the interpolation nodes, multiplied by the least deviation. Note for a given interpolation T, the bound $\|\mathcal{L}_T\|_p d_n(f)$ for $\|\mathcal{L}_T(w_n - f)\|_p$ is optimal, *i.e.*, there exists a nonpolynomial continuous function f such that $\|\mathcal{L}_T\|_p d_n(f) = \|\mathcal{L}_T(w_n - f)\|_p$, where w_n is the best approximating polynomial in Theorem 3. In this sense, we say the interpolation error bound in Theorem 6 is optimal.

We shall study the efficacy of the interpolating polynomials as the number of interpolation nodes increases. We shall consider an infinite triangle array consisting of $T^1, ..., T^n, ...$, where $T^i = (\tau_0^{(i)}, \tau_1^{(i)}, ..., \tau_i^{(i)})$. $\{T^i\}$ defines an interpolation scheme. Given f, What is the convergence property of $\mathcal{L}_{T^i} f$?

Theorem 7 [7] For any $\{ T^i \}$, $\|\mathcal{L}_{T^n}\|_\infty \geq 2\pi^{-2} log(n+1) + b_n$, where b_n is a bounded sequence.

The right hand side of this inequality is related to the norm of the Fourier convolution operator given by the Dirichlet kernel.

Theorem 8 (Erdos) [3] There exists a positive constant c such that for any $\{ T^i \}$, $\|\mathcal{L}_{T^n}\|_\infty \geq 2\pi^{-1} log(n+1) - c$.

It is possible to deduce the following result from Theorem 7 or Theorem 8,

Theorem 9 (Faber) [7] No matter what are the nodes of an interpolation scheme $\{ T^i \}$, $i \geq 1$, there is a continuous function $f(\cdot)$ such that the sequence of interpolating polynomials relative to the given scheme diverges in L^∞ norm.

It is important to realize that such functions are non-differentiable in I, in fact, they are not even piecewise differentiable. On the other hand, for a given continuous function, it is always possible to choose an interpolation scheme (depending on the function) such that the interpolating polynomials converge to the function in L^∞ norm.

To illustrate the importance of the choice of the interpolation nodes, consider the uniform interpolation scheme $T^n_{unif} = \{-1 + 2i/n, i = 0, 1, ..., n\}$, then

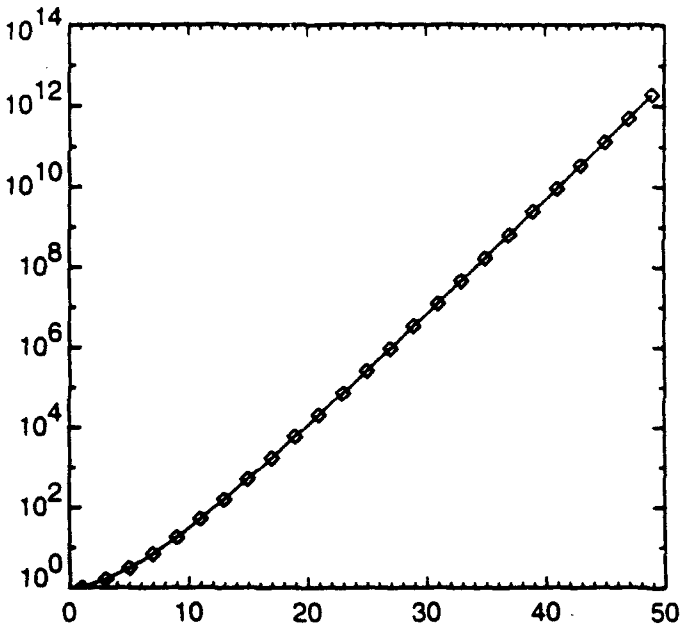

Figure 1: $\lambda(T^n_{unif})$ as a function of n.

Theorem 10 [7] For a uniform system of nodes T^n_{unif}, $\lambda(T^n_{unif}) \geq Cexp(n/2)$, where $C > 0$.

The exponential divergence of $\lambda(T^n_{unif})$ is apparent in Fig. 1. Theorem 10 implies that $\|\mathcal{L}_{T^n_{unif}}\|_p$ diverges at least exponentially for $p \geq 1$. This follows from the following lemma.

Lemma 1 [5]: Let u be a polynomial of degree n, then for $1 \leq p \leq q \leq \infty$, $\|u\|_q \leq (2n)^{1/p-1/q}\|u\|_p$.

Take $q = \infty$ in Lemma 1, it follows that for an n-th order interpolation operator $\|\mathcal{L}_T\|_p \geq \|\mathcal{L}_T\|_\infty/(2n)^{1/p}$.

Note that from theorem 6, $f - \mathcal{L}_T f$ converges in L^p norm for the uniform interpolation scheme if $d_n(f)$ decreases faster than $\|\mathcal{L}_{T^n_{unif}}\|_p$. However, this is not the case for functions with only finite number of derivatives. Even for analytic (or infinitely

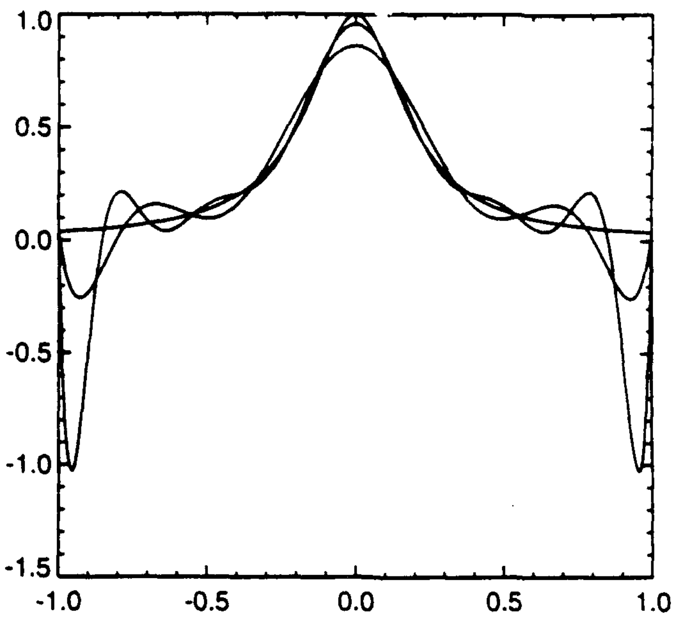

Figure 2: The function $R(t) = (1+25t^2)^{-1}$ and $\mathcal{L}_{T^9_{unif}}R$, $\mathcal{L}_{T^{13}_{unif}}R$.

differentiable) functions, this condition may not be satisfied. As Runge showed that for $R(t) = (1+25t^2)^{-1}$, $\mathcal{L}_{T^n_{unif}}R$ diverges from R in intervals $0.728 < |t| < 1$. This is illustrated in Fig. 2 for $\mathcal{L}_{T^9_{unif}}R$ and $\mathcal{L}_{T^{13}_{unif}}R$. We see that as interpolation order n increases, the interpolant $\mathcal{L}_{T^n_{unif}}R(t)$ diverges farther away from $R(t)$ in the specified intervals. In this sense, the uniform interpolation scheme is generally considered "not good". Bernstein gave a similar result for the piecewise differentiable function $B(t) = |t|$, He showed that $\mathcal{L}_{T^n_{unif}}B(t)$ diverges from $B(t)$ in intervals $0 < |t| < 1$.

For T consists of the zeroes of the $(n+1)$-th order Chebyshev polynomials $T^n_{Cheb} = \{-cos\frac{(2i+1)\pi}{2(n+1)}, i = 0, ..., n\}$, we have

Theorem 11 (Chebyshev) [6] $\lambda(T^n_{Cheb}) \le 2\pi^{-1}log(n+1) + 4\pi^{-1}(1+log4)$.

In view of Theorem 8, we see that the Chebyshev interpolation scheme $\{\ T^n_{Cheb}\ \}$ is almost optimal.

Note that other than in L^∞ norm, under an appropriate choice of interpolation scheme $\{ T^i \}$, $\mathcal{L}_{T^i} f$ may converge to f for all $f \in C(I)$. In particular, if we choose T to be the quadrature points for the Gauss-Legendre quadrature, *i.e.*, $T^n_{Leg} := (x_1, x_2, ..., x_{n+1})$, where x_k's are the zeroes of the $(n+1)$-th order Legendre polynomial, $P_{n+1}(t)$, then

Theorem 12 [6] If $f(\cdot) \in C(I)$, $\|f - \mathcal{L}_{T^n_{Leg}} f\|_2 \leq 2\sqrt{2} d_n(f)$; $\|f - \mathcal{L}_{T^n_{Leg}} f\|_1 \leq 4 d_n(f)$;

Similarly, we can also choose T to be the quadrature points for the Gauss-Lobatto quadrature, *i.e.*, $T^n_{Lob} = (-1, x'_1, ..., x'_{n-1}, 1)$, where x'_k's are the zeroes of the derivative of the n-th order Legendre polynomial, $P'_n(t)$. The same inequalities hold for $f - \mathcal{L}_{T^n_{Lob}} f$. We note that for T^n_{Cheb}, similar inequalities hold.

The nodes for the Fejer interpolation scheme consists of the zeroes of the integral of the Legendre polynomials, *i.e.*, $T^n_{Fejer} = (x^\dagger_1, x^\dagger_2, ..., x^\dagger_{n+1})$, where $x^\dagger_k$'s are the zeroes of the integral of the n-th order Legendre polynomial, $\int_{-1}^t P_n(x)dx$. Note that $x^\dagger_1 = -1, x^\dagger_{n+1} = 1$.

Interpolation schemes using $\{ T^n_{Leg} \}$, $\{ T^n_{Lob} \}$, $\{ T^n_{Fejer} \}$, $\{ T^n_{Cheb} \}$ will be called Legendre, Lobatto, Fejer and Chebyshev schemes respectively.

Maday has recently shown that if $f(\cdot) \in H^m(I)$, *i.e.*, if all the derivatives of f up to order m are square integrable, then $\mathcal{L}_{T^n_{Lob}} f$ converges to f in $H^1(I)$ if $m > 1$:

Theorem 13 (Maday) [1] If $f \in H^m(I)$, then there is a positive constant c, such that $\|f - \mathcal{L}_{T^n_{Lob}} f\|_{H^1(I)} \leq c n^{1-m} \|f\|_{H^m(I)}$.

The H^m norm for f is $\|f\|_{H^m} = (\int_{-1}^1 \sum_{i=0}^m [\partial^i f(t)]^2 dt)^{1/2}$, where $\partial^i f(t)$ denotes the i-th derivative of f.

Erdos has shown the following results concerning the L^1 pseudonorm ($|L|^1$ norm) and the mean norm for the interpolation operator

Theorem 14 (Erdos) [3] There exists a positive constant C such that for any n-th order interpolation, $|\mathcal{L}_T|_1 > Clog(n+1)$.

Theorem 15 (Erdos) [3] To every positive ϵ, there exists a positive integer N such that for every n-th order interpolation operator T satisfying $n > N$, $\|\langle \mathcal{L}_T \rangle\|^2 > 2 - \epsilon$

By definition of the Gauss-Legendre quadrature, we have for the Legendre interpolation scheme, $\|\langle \mathcal{L}_{T^n_{Leg}} \rangle\|^2 = 2$.

Theorem 16 (Fejer) [4] The only n-th order interpolation set for which $\sum_{k=0}^{n} |L_k(t)|^2 \leq 1, t \in I$ is the n-th order Fejer set T^n_{Fejer}.

Note that T^n_{Lob} and T^n_{Fejer} include the end points {-1,1} of I. Hence, if a function is piecewise approximated by polynomial interpolation in contiguous intervals using the Lobatto sets or the Fejer sets, its continuity at the joint points is guaranteed. This property is essential for the continuity of the piecewise interpolated polynomial function in an interval.

III. Minimal Interpolation Sets

In the following, we consider interpolation schemes whose nodes include the end points {-1,1}. From Theorem 6, for a given interpolation T, the optimal upper bound for the interpolation error of a function f in L^∞ norm is $(1 + \|\mathcal{L}_T\|_\infty)d_n(f)$. We can minimize this optimal error upper bound by minimizing the Lebesgue constant of the interpolation operator $\|\mathcal{L}_T\|_\infty$ through the redistribution of the interpolation nodes (the least deviation $d_n(f)$ by definition is already the smallest possible error).

Similarly, the interpolation error of a function f in L^p norm is less than $(2^{1/p} + \|\mathcal{L}_T\|_p)d_n(f)$. However, $\|\mathcal{L}_T\|_p$ is hard to work with. Since $\|\mathcal{L}_T\|_r < |\mathcal{L}_T|_p$, $(2^{1/p} + |\mathcal{L}_T|_p)d_n(f)$ also gives an upper bound for the interpolation error in L^p norm, we can instead minimize $|\mathcal{L}_T|_p$ which has a closed form. We shall see that the corresponding minimal sets are also good interpolation sets. We shall say that an interpolation set is minimal in $|L|^p$ or L^∞ if it minimizes $|\mathcal{L}_T|_p$ or $\|\mathcal{L}_T\|_\infty$. We shall say an interpolation set is minimal in the mean if it minimizes $\|\langle\mathcal{L}_T\rangle\|$. n-th order minimal sets are denoted as $T^n_{L^\infty}$, $T^n_{|L|^p}$ and $T^n_{\langle L\rangle}$ respectively. Interpolation schemes using $\{ T^n_{|L|^p} \}$, $\{ T^n_{L^\infty} \}$ and $\{ T^n_{\langle L\rangle} \}$ are called minimal interpolation schemes in $|L|^p$, L^∞ and in the mean. We shall see that minimal interpolation sets have quite similar properties.

i. Minimal interpolation sets in L^∞

For a given system of interpolation nodes $T = (\tau_0, \tau_1, ..., \tau_n)$, where $\tau_0 = -1, \tau_n = 1$, let $\lambda_i(T) = max_{t\in[\tau_{i-1},\tau_i]} \sum_{k=0}^n |L_k(t)|$, then $\lambda(T) = max_{i=1,...,n}\lambda_i(T)$.

Conjecture 1 T minimizes $\|\mathcal{L}_T\|_\infty$ if and only if $\lambda_1(T) = \lambda_2(T) = ... = \lambda_n(T)$, these $n-1$ conditions uniquely determine the $n-1$ interpolation nodes $\tau_1, ..., \tau_{n-1}$.

Note that the condition that the minimal sets contain the end points {-1,1} of the interval I is essential. The minimal sets are not unique if {-1,1} is not included in the minimal sets. As can be verified in the case of order 2 interpolation, any interpolation set $T^2_y = (-y, 0, y)$ where $2\sqrt{2}/3 \le y \le 1$ is also minimal.

The uniqueness implies that the minimal sets are symmetrical, *i.e.* they are invariant under the reflection $x \to -x$.

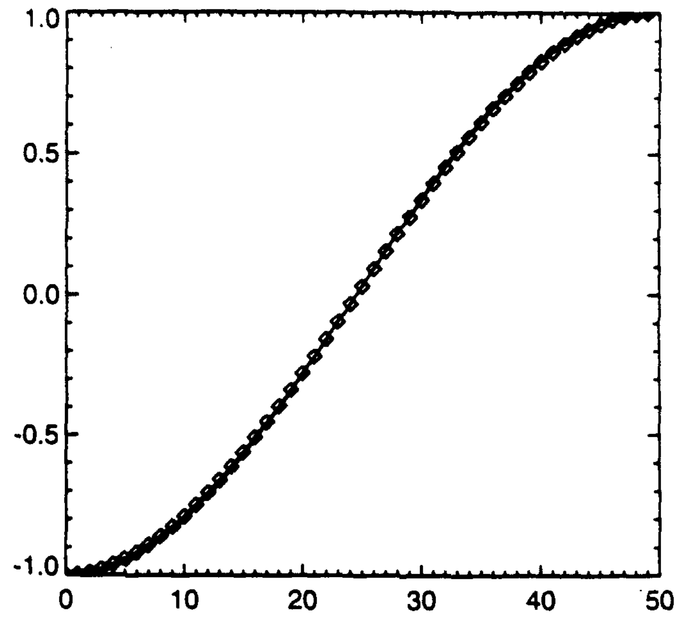

Figure 3: The coordinates for $\mathcal{T}^{49}_{L^\infty}$.

We have numerically found the sets satisfying the equalities in Conjecture 1 for interpolation order up to $n = 50$. They appear to be minimal in L^∞. We shall denote n-th order such set by $\mathcal{T}^n_{L^\infty}$. Conjecture 1 states that $T^n_{L^\infty} = \mathcal{T}^n_{L^\infty}$.

In the appendix, we list the coordinates for $\mathcal{T}^n_{L^\infty}$ from $n = 1$ to $n = 20$. The error in the coordinate is less than 10^{-7}. Fig. 3 shows the coordinates of $\mathcal{T}^{49}_{L^\infty}$. They are almost uniformly distributed in the circle coordinate: $\theta = -cos^{-1}(x)$,

Asymptotically $\lambda(\mathcal{T}^n_{L^\infty})$ scales with $log(n+1)$. Fig. 4 shows the Lebesgue constant (the L^∞ norm) for $\mathcal{T}^n_{L^\infty}$. Our numerical result gives asymptotically $\lambda(\mathcal{T}^n_{L^\infty}) \rightarrow 2\pi^{-1}log(n+1) + 0.522 \sim 0.523$. This is consistent with Theorem 8. We note that even at order 50 (which is considerably higher than the interpolation order in any practical situation), $\lambda(\mathcal{T}^n_{L^\infty})$ is only approximately 3, hence, from Theorem 6, the interpolation error would be at most 4 times the least deviation at order 50. Note the Lebesgue

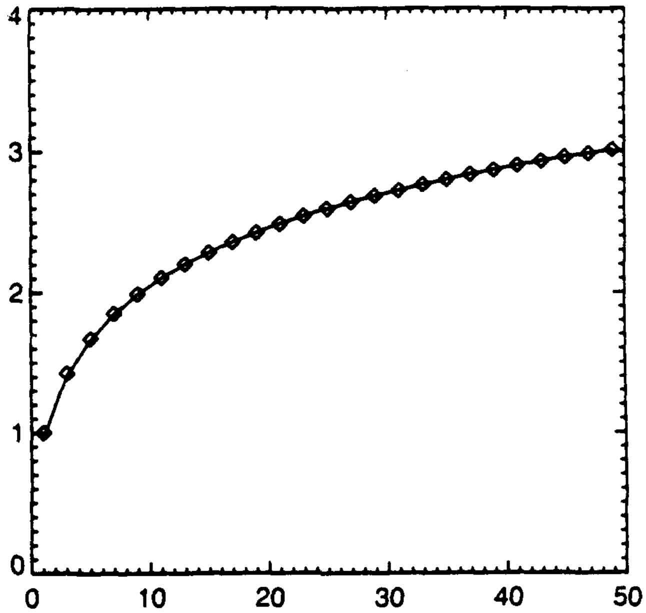

Figure 4: $\lambda(\mathcal{T}^n_{L^\infty})$ as a function of n.

constant of the interpolation operator depends very sensitively on the distribution of the interpolation nodes. For the uniform distribution, from Fig. 1, the Lebesgue constant at order 20 already exceeds 10^4.

The $|L|^p$ norm of $\mathcal{T}^n_{L^\infty}$ also appears to scale with $log(n+1)$. However, the square of the mean norm saturates from below at 2.

ii. Minimal interpolation sets in $|L|^1$, $|L|^2$ and in the mean

The minimal interpolation sets in $|L|^1$, $|L|^2$ and in the mean are found by using some standard minimization procedure for the corresponding norms.

We shall illustrate in this section that $T^n_{|L|^1}$, $T^n_{|L|^2}$, $T^n_{\langle L\rangle}$ have similar scaling behaviors as $\mathcal{T}^n_{L^\infty}$. Their coordinate positions are relatively close, and their $|L|^p$ norms, which determine the upper bounds for the interpolation error, are almost the same.

It was conjectured [3] that the n-th order minimal set for the mean norm is T^n_{Fejer},

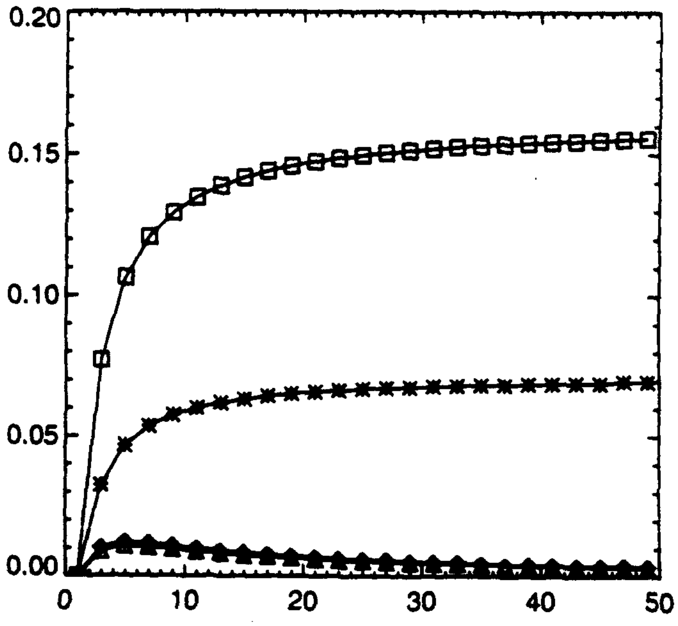

Figure 5: the differences in the Lebesgue constant (the L^∞ norm) between $T^n_{|L|^1}$ and $T^n_{L^\infty}$ (diamonds), $T^n_{|L|^2}$ and $T^n_{L^\infty}$ (triangles), $T^n_{(L)}$ and $T^n_{L^\infty}$ (asterisks), T^n_{Lob} and $T^n_{L^\infty}$ (squares) respectively.

the n-th order Fejer set. The mean minimal sets we have found indicate that this conjecture is incorrect.

Fig. 5 shows the differences in the Lebesgue constant (L^∞ norm) between $T^n_{|L|^1}$ and $T^n_{L^\infty}$, $T^n_{|L|^2}$ and $T^n_{L^\infty}$, $T^n_{(L)}$ and $T^n_{L^\infty}$, T^n_{Lob} and $T^n_{L^\infty}$ respectively. These differences appear to asymptote to constants. In the case of $T^n_{|L|^1}$ and $T^n_{|L|^2}$, it appears that they are very close to $T^n_{L^\infty}$ as n increases (the asymptotic differences in the Lebesgue constants are close to zero). The differences in the Lebesgue constants between all these sets are small so that these sets are almost the same in practical interpolation situations. *E.g.*, if we use $T^n_{(L)}$ instead of $T^n_{L^\infty}$ the difference in the Lebesgue constant (L^∞ norm) is only of order 0.07. According to theorem 6, this only increases the interpolation error for approximately 0.07 times the least deviation, which is usually

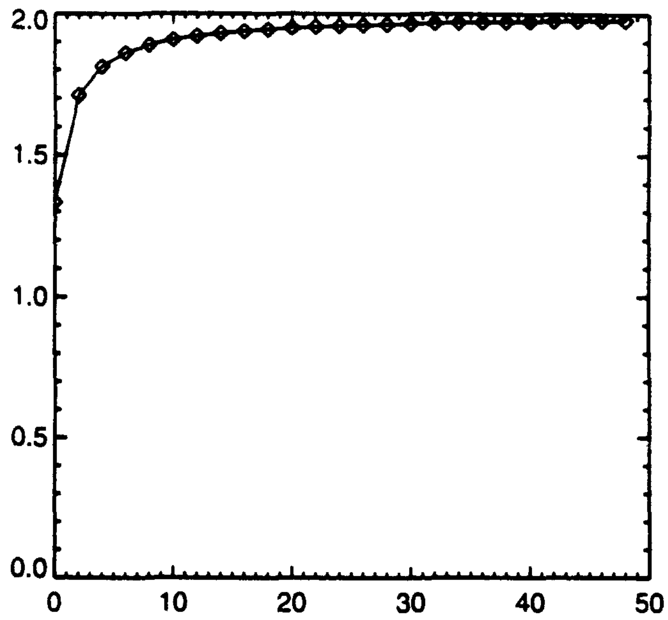

Figure 6: The mean norm for the mean minimal set $\|\langle \mathcal{L}_{T^n_{\langle L \rangle}} \rangle\|^2$ as a function of n.

negligible in practice.

The L^∞ norm as well as the $|L|^p$ norm for all minimal sets scale with $log(n+1)$. The square of the mean norm, $\|\langle \mathcal{L}_T \rangle\|^2$, asymptotes to 2 from below for all minimal sets. Fig. 6 shows the dependence of the mean norm for the mean minimal set $T^n_{\langle L \rangle}$ on the interpolation order n.

iii. The Lobatto sets

For the Lobatto sets, using the Gauss-Lobatto quadrature formula, one obtains that the square of the mean norm $\|\langle \mathcal{L}_{T^n_{Lob}} \rangle\|^2$ equals to $2 - C_n$, where $C_n > 0$ and $lim_{n \to \infty} C_n = 0$. From Theorem 15, one sees that the Lobatto sets are almost optimal in the mean norm.

The Lobatto sets are usually good enough for interpolation purpose in practice. (As is any set consisting of the zeroes of any other orthogonal polynomial in I.)

However, albeit there are generalization of orthogonal polynomials in some standard connected domains in several dimensions, their relations to quadrature and interpolation are unclear. The minimal sets, on the other hand, have direct generalization in these situations and in view of Theorem 6, they are the "optimal" interpolation sets [2].

IV. Approximation of Derivatives of Functions

Frequently, we not only need to approximate the function, but also its derivative.

Consider the following problem. Suppose we have only access to the function values, but not its derivative values at certain points. we want to extrapolate information about its derivative. Theorem 1 again provides a way to *estimate* the derivative of the function by evaluating the derivative of the interpolating polynomial:

$p_n'(t) = \mathcal{L}_T' f(t) := \sum_{k=0}^{n} f(\tau_k) L_k'(t)$

Here prime ($'$) denotes taking derivative with respect to t. We call $\mathcal{L}_T'$ the derivative interpolation operator. We note that it is not a projection operator.

Let q_n be an n-th order polynomial, using the method of intermediate approximation, $\|f' - \mathcal{L}_T' f\|_p \leq 2^{1/p}\|f' - q_n'\|_\infty + \|\mathcal{L}_T'\|_p \|f - q_n\|_\infty$. where

$$\|\mathcal{L}_T'\|_\infty = \lambda'(T) := sup_{t\in I} \sum_{k=0}^{n} |L_k'(t)|$$

$$\|\mathcal{L}_T'\|_p \leq |\mathcal{L}_T'|_p := (\int_{-1}^{1} (\sum_{k=0}^{n} |L_k'(t)|)^p dt)^{1/p}.$$

Again we shall call $|\mathcal{L}_T'|_p$ the L^p pseudonorm or the $|L^p|$ norm of the derivative operator and $\lambda'(T)$ the Lebesgue constant for the derivative operator.

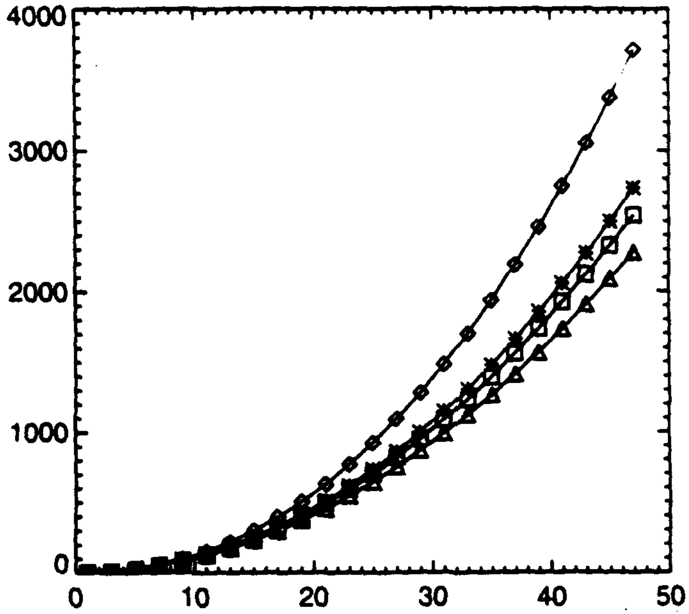

Figure 7: The Lebesgue constants for the derivative interpolation operator for $T^n_{|L'|^2}$ (triangles), $T^n_{\langle L'\rangle}$(squares), $T^n_{L^\infty}$(asterisks) and T^n_{Lob}(diamonds).

The approach we have used in the previous section can be used to minimize $|\mathcal{L}'_T|_p$. We shall minimize $|\mathcal{L}'_T|_p$ and the mean norm for the derivative interpolation operator $\|\langle\mathcal{L}'_T\rangle\| := (\int_{-1}^{1}\sum_{k=0}^{n}|L'_k(t)|^2 dt)^{1/2}$. We shall denote the corresponding minimal sets by $T^n_{|L'|^p}$ and $T^n_{\langle L'\rangle}$ respectively.

The error for the interpolation of the derivative is usually largest at the end points. In quite contrast to the Lebesgue constant for the interpolation operator, the Lebesgue constant for the derivative interpolation operator $\lambda'(T)$ appears to increase quadratically with n for both $T^n_{|L'|^p}$ and $T^n_{|L|^p}$. Fig. 7 shows this scaling of the Lebesgue constant for the derivative interpolation operator for $T^n_{|L'|^2}$, $T^n_{\langle L'\rangle}$, $T^n_{L^\infty}$ and T^n_{Lob}.

However, the $|L|^p$ and the mean norm for the derivative interpolation operator appear to increase linearly with the interpolation order n. As is illustrated in Fig. 8 for the $|L|^2$ norm. This is a result similar to Theorem 13.

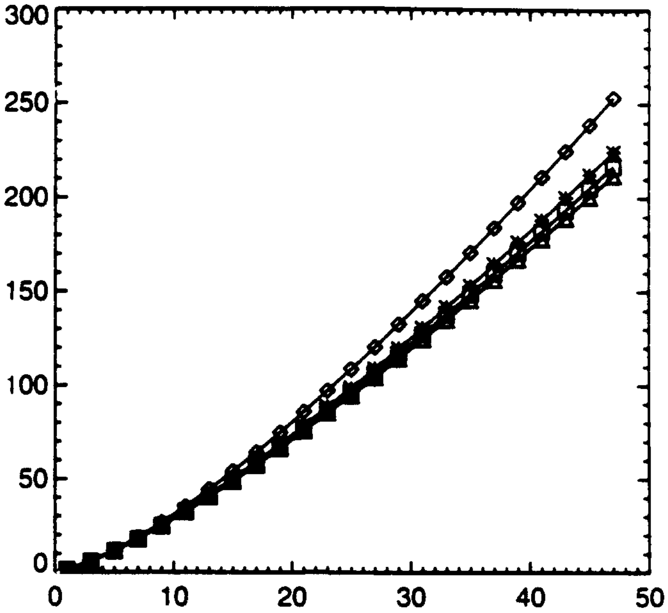

Figure 8: The $|L|^2$ norms for the derivative interpolation operator for $T^n_{|L'|^2}$ (triangles), $T^n_{\langle L' \rangle}$(squares), $T^n_{L\infty}$(asterisks) and T^n_{Lob}(diamonds).

We note that the values for the Lebesgue constants for the derivative interpolation operator for different minimal sets are quite close. This indicates that the minimal sets for the derivative interpolation operator are quite close to the minimal sets for the interpolation operator.

Suppose we are not only able to access the function value, but also its derivative at certain n+1 points. Then we can use Hermite interpolation formula to obtain a $2n+1$ order polynomial to approximate the function:

$p_{2n+1}(t) = \mathcal{H}_T f(t) := \sum_{k=0}^{n}(f(\tau_k)H_k(t) + f'(\tau_k)\tilde{H}_k(t))$, where $H_k(t) = (1 - 2L_k'(\tau_k)(t-\tau_k))L_k^2(t)$, $\tilde{H}_k(t) = (t-\tau_k)L_k^2(t)$.

The minimal sets for the Hermite interpolation operator $\mathcal{H}_T$ can be considered similarly.

V. Examples

Let us consider the interpolation of the following rational function $R_\beta(t) = \frac{1}{1+(\beta t)^2}$. In Fig. 9, we show the interpolation error $R_1(t) - \mathcal{L}_T R_1(t)$ for minimal sets $T^n_{|L|^2}$, $T^n_{|L|^1}$, $T^n_{L^\infty}$, $T^n_{\langle L\rangle}$ and the Lobatto set T^n_{Lob} for order $n = 9$ interpolation. Interpolation using minimal sets does appear to have smaller interpolation error than any known interpolation schemes. The magnitudes of the interpolation error for different minimal sets are of the same order. Indeed, $T^n_{L^\infty}$ appears to have the smallest interpolation error and T^n_{Lob} has the largest error while the interpolation error for $T^n_{\langle L\rangle}$ seems to lie in between. The interpolation errors for $T^n_{|L|^2}$ and $T^n_{|L|^1}$ appear to be very close to that of $T^n_{L^\infty}$. This is consistent with the fact that the norms for $T^n_{|L|^2}$ and $T^n_{|L|^1}$ are very close to the norm of $T^n_{L^\infty}$. The ordering of the L^∞ norms for minimal sets appear to be indicative of the ordering of their interpolation errors.

On the other hand, if we interpolate the same function to extrapolate the derivative, the error is larger. We show in Fig. 10 the derivative error $R_1'(t) - \mathcal{L}'_T R_1(t)$ for minimal sets $T^n_{L'^2}$, $T^n_{\langle L'\rangle}$, $T^n_{L^\infty}$ and T^n_{Lob} for interpolation order $n = 9$. The error in derivative is largest at the end points $t = \pm 1$. $T^n_{L'^2}$ appears to have the smallest derivative error, while $T^n_{L^\infty}$ has the largest error. They all have the same order in interpolation error.

For minimal sets, both the interpolation error in L^2 norm or in H^1 norm for R_β decreases exponentially (see Fig. 11) , in accordance with Theorem 12 and the fact that the function R_β is analytic hence $d_n(R_\beta)$ decreases exponentially. The interpolation error in H^1 norm is dominated by the error in its derivative, since as

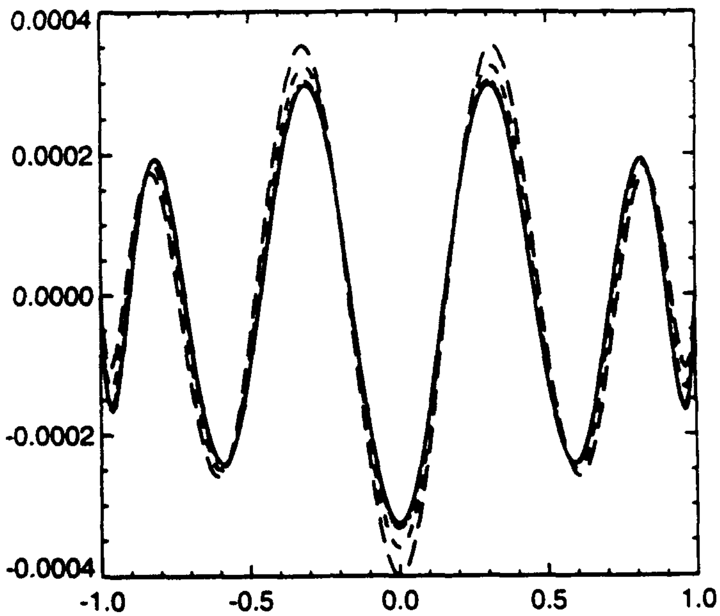

Figure 9: The interpolation error $R_1(t) - \mathcal{L}_T R_1(t)$ for minimal sets $T^n_{|L|^2}$ (dot), $T^n_{|L|^1}$ (dash dot), $T^n_{L^\infty}$(solid), $T^n_{\langle L\rangle}$ (dash)and the Lobatto set T^n_{Lob} (long dash) for order $n = 9$.

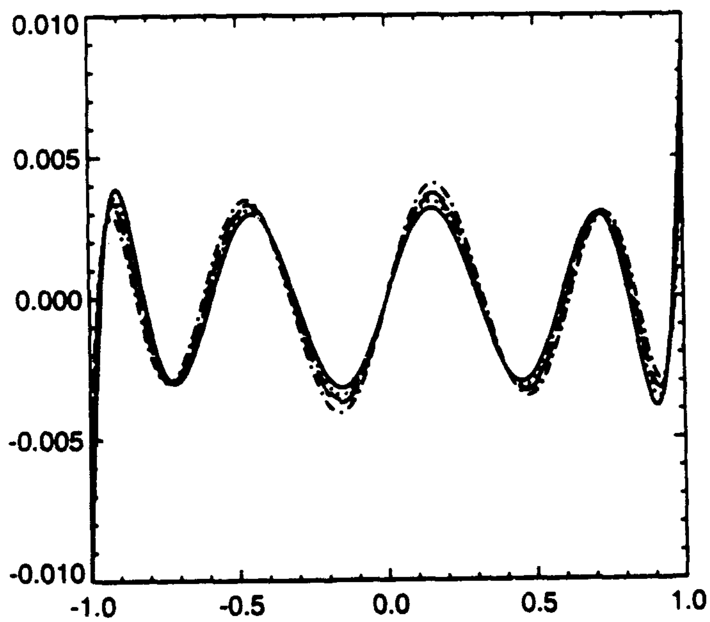

Figure 10: The derivative error $R_1'(t) - \mathcal{L}'_T R_1(t)$ for minimal sets $T^n_{L'^2}$ (dash), $T^n_{\langle L'\rangle}$ (dot), $T^n_{L^\infty}$ (solid) and T^n_{Lob} (dash dot) for interpolation order $n = 9$.

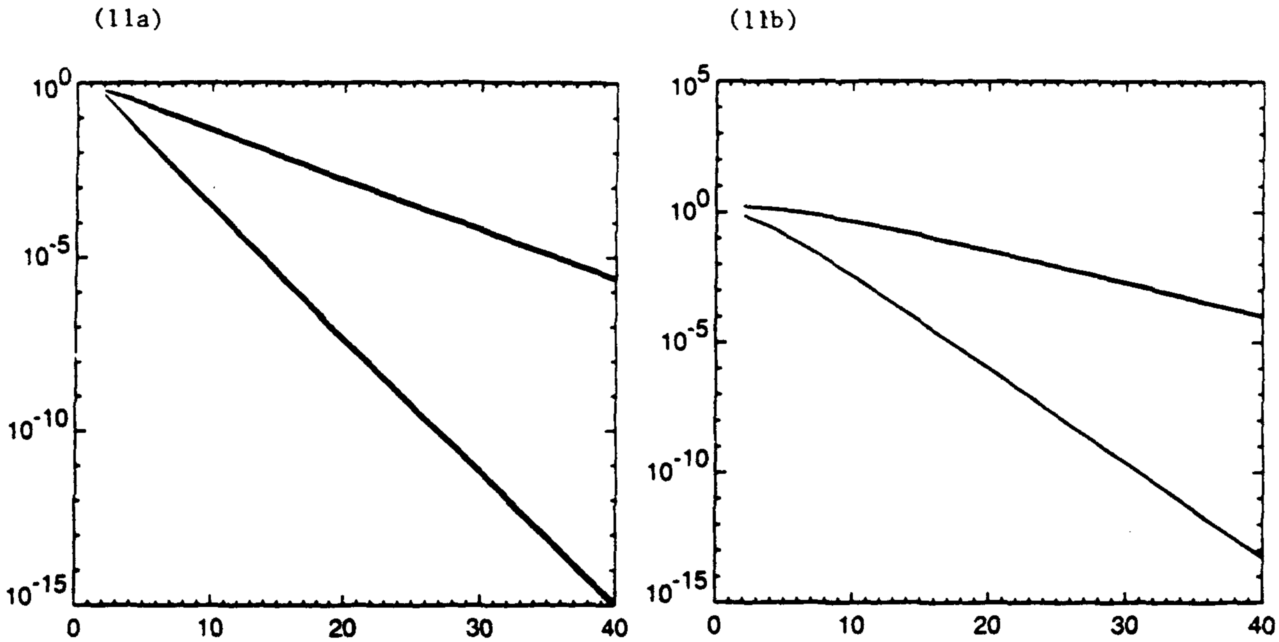

Figure 11: The interpolation error in L^2 norm (11a) and H^1 norm (11b) for R_β where $\beta = 1$ and $\beta = 3$.

we have seen, the error in the derivative of the function from interpolation is much larger than the error in the function itself. All minimal sets seems to have almost the same order of interpolation error in L^2 and H^1 norms.

VI. Adaptive Determination of Approximation Order

Suppose we want to approximate a function in an interval. It is neither appropriate nor practical to use polynomial interpolation for the function in the entire interval.

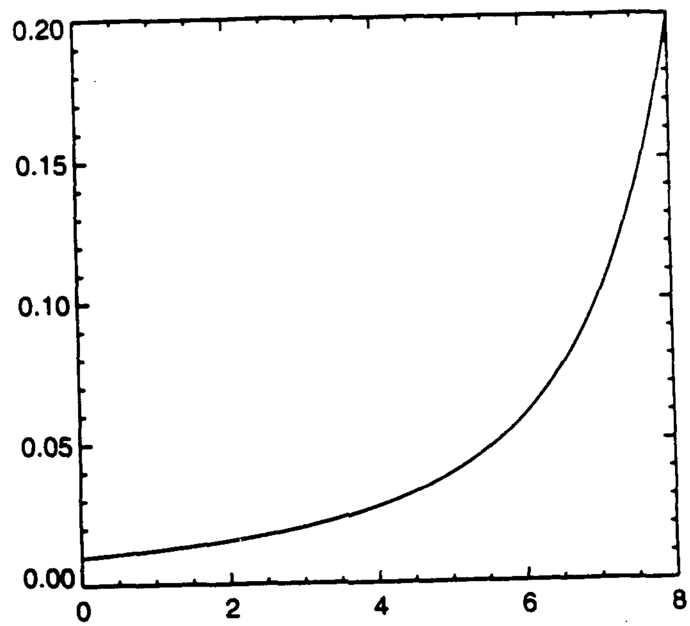

Figure 12: The polynomial interpolation of the function $f = \frac{1}{(x-10)^2+1}$ in 4 subintervals $I_1 = [0, 2], I_2 = [2, 4], I_3 = [4, 6], I_4 = [6, 8]$

.

Approximation of functions using polynomials of arbitrarily large order may be prohibitively expensive. This problem could be solved by dividing the whole interval into subintervals and by successively approximating the function in each individual subinterval. Note that if we use the minimal sets discussed in section III which contains the end points of the interval, the continuity of the function at junction points is automatically guaranteed.

However, if we use the same polynomial order for interpolation in all the subintervals, the approximation error may be of quite different order of magnitude in different subintervals. For example, in Fig. 12. we show the polynomial interpolation of the function $f = \frac{1}{(x-10)^2+1}$ in 4 subintervals $I_1 = [0, 2], I_2 = [2, 4], I_3 = [4, 6], I_4 = [6, 8]$ using the same order 9 polynomial interpolation. We see that the function is well

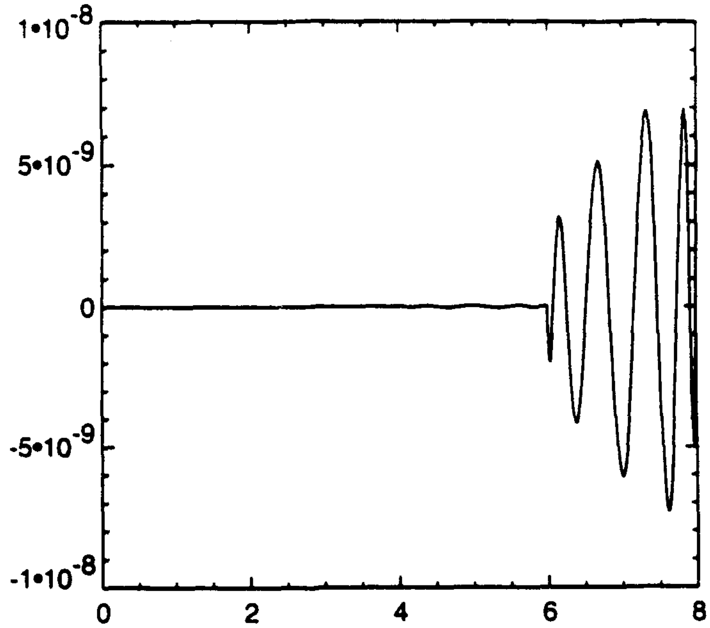

Figure 13: The error $f - \mathcal{L}_T f$ in different subintervals for order 9 interpolation.

approximated in each subinterval, however, as is shown in Fig. 13, the error $f - \mathcal{L}_T f$ varies with quite different order of magnitude in different subintervals. In fact, compared to the error in I_4, errors in other subintervals are negligible. Frequently, it is inefficient to have such a situation, since both the access of the function values at interpolation nodes and the interpolation procedure cost more for high order interpolation than for low order interpolation. The interpolation order in subintervals I_1, I_2, I_3 can in fact be reduced to achieve the same order of accuracy (10^{-8} in all subintervals). As is illustrated in Fig. 14, to achieve the same order of accuracy for the approximation, we only need order 5 interpolation in I_1, order 6 in I_2, and order 7 in I_3.

In the following, we give a simple adaptive procedure for the determination of the approximation order. Let $\{ T^i = (\tau_0^{(i)}, \tau_1^{(i)}, ..., \tau_i^{(i)}), i = 0, ..., n, ...; \tau_0^{(i)} = -1, \tau_i^{(i)} = 1, \tau_0^{(0)} = 0 \}$ be the interpolation scheme in $I = [-1, 1]$. Since low order interpolation

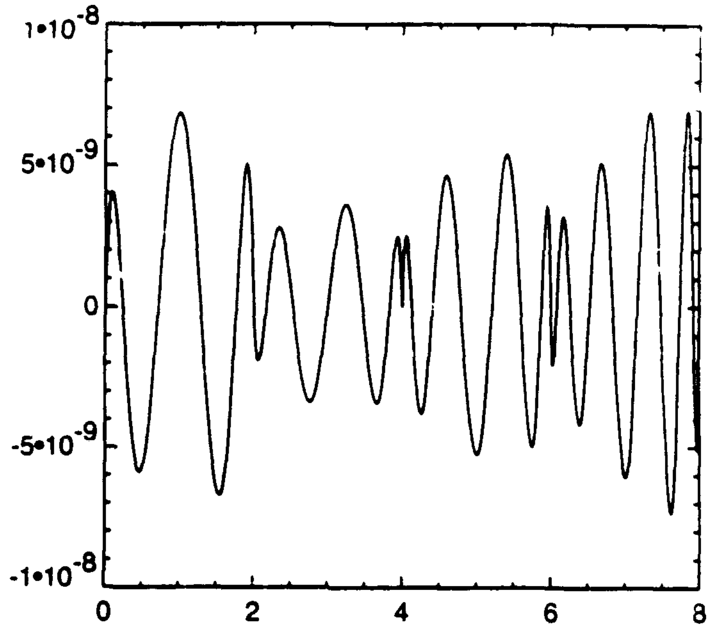

Figure 14: The error $f - \mathcal{L}_T f$ in different subintervals for interpolati n using adaptively determined order.

informations are relatively cheap to obtain, the interpolation error can be estimated by using low order quantities. At order n, we have at our disposal quantities $\{ f(\tau_j^{(i)}), j = 0, ..., i; i = 1, ..., n \}$. Define the estimated interpolation error at order n by $e_n(f) = max_{j=0,...,i;i=0,...,n-1} |f(\tau_j^{(i)}) - (\mathcal{L}_{T^n} f)(\tau_j^{(i)})|$. For a prescribed error tolerance $\rho(f)$, calculate the estimated interpolation error $e_n(f)$. If $e_n(f) < \rho(f)$, stop; otherwise, continue to go to higher order polynomial interpolation until $e_n(f) < \rho(f)$.

Fig. 14 is generated using this adaptive procedurc. We see that it consistently gives the appropriate interpolation order for different subintervals.

Numerical computation indicates that for all the minimal sets we have considered as well as the Lobatto interpolation schemes, the above error estimator satisfies $\frac{e_n(f)}{\|f - \mathcal{L}_{T^n} f\|_\infty} \to 1$, *i.e.*, $e_n(f)$ is an asymptotic error estimator for these sets. This is because all these sets are dense in I.

References

[1] Bernardi, C. and Maday, Y.(1992). Approximations Spectrales de Problèmes aux Limites Elliptiques. Springer-Verlag, Paris.

[2] Chen, Q. and Babuška, I. (1993). Polynomial Interpolation of Real Functions II, Symmetrical Polynomial Interpolation for the Trangle.

[3] Erdos, P. (1961). Problems and Results on the Theory of Interpolation. II. *Acta Math. Acad. Sci. Hungar.* 12, pp235-244.

[4] Fejer, L. (1932). Bestimmung derjenigen Abszissen eines Intervalles, fur wilche die Quadratsumme der Grundfuncionen der Lagrangeshen Interpolation im Intervelle [-1,+1] ein moglichst kleines Maximum besitzt, *Annali della Scuola Norm. Sup. di Pisa* (2), 1, pp. 3-16.

[5] Quarderoni, A. (1984). Some Results of Bernstein and Jackson Type for Polynomial Approximation in L^p-space. *Japan J. Appl. Math.* 1, pp176-181.

[6] Rivlin, T.J. (1969). An Introduction to the Approximation of Functions. Dover, New York.

[7] Tikhomirov, V.M. (1990). Approximation Theory, in Analysis II, Gamkrelidze, R.V. Ed., Encyclopaedia of Mathematical Sciences, vol 14. Springer-Verlag, Berlin.

Appendix

The coordinates of $\mathcal{T}_{L^\infty}^n$ Minimal Sets. Minimal sets are symmetrical under the reflection $x \to -x$. Only nonnegative coordinates are given.

n = 1

1.000000000000000

n = 2

0.000000000000000 1.000000000000000

n = 3

0.4177912974318541 1.000000000000000

n = 4

0.0000000000000000 0.6209113447507076

1.000000000000000

n = 5

0.2689070754306328 0.7341271205122524

1.000000000000000

n = 6

0.0000000000000000 0.4461215875027044

0.8034402333430620 1.000000000000000

n = 7

0.1992876906268561 0.5674310617087234

0.8488719620155624 1.000000000000000

n = 8

0.0000000000000000 0.3477880851677845

0.6535334678686619 0.8802308501922200

1.000000000000000

n = 9

0.1585652002417824 0.4601504943340195

0.7166138903070984 0.9027709837119849

1.000000000000000

n = 10

0.0000000000000000 0.2848881583261380

0.5466676817797121 0.7640984500324875

0.9195087505000881 1.000000000000000

n = 11

0.1317517498523124 0.3862689500044784

0.6144355686309110 0.8006822718709737

0.9322747835437754 1.000000000000000

n = 12

0.0000000000000000 0.2412238223627454

0.4684174854953188 0.6683666122414075

0.8294354770322080 0.9422316270662717

1.000000000000000

n = 13

0.1127326359341456 0.3325422077453501

0.5356655185494467 0.7119103291764621

0.8524275965461744 0.9501460629121580

1.000000000000000

n = 14

0.0000000000000000 0.2091512677285655

0.4091565197775405 0.5912705391349697

0.7475281136999568 0.8710916050399280

0.9565402629254455 1.000000000000000

n = 15

0.0985297784103952 0.2918018772197364

0.4738547252988244 0.6376896893581207

0.7770061975121303 0.8864437448634451

0.9617797393250818 1.000000000000000

n = 16

0.0000000000000000 0.1845994015541723

0.3629096447556914 0.5288572825810849

0.6767882746653493 0.8016617880341823

0.8992200395501156 0.9661264747628877

1.000000000000000

n = 17

0.0875146300475834 0.2598845240957428

0.4243549140109628 0.5759276777085162

0.7099951825658469 0.8224813031663734

0.9099637721447408 0.9697722157216912

1.000000000000000

n = 18

0.0000000000000000 0.1652022750114381

0.3258963792565789 0.4776989264800610

0.6164674647713962 0.7384152647561639

0.8402138563369347 0.9190827135791494

0.9728598817268702 1.000000000000000

n = 19

0.0787199876854192 0.2342217468993406

0.3839541966628281 0.5242305008694372

0.6515953463715364 0.7629113934252203

0.8554359783038173 0.9268876580527369

0.9754977712411774 1.000000000000000

n = 20

0.0000000000000000 0.1494917070738174

0.2956424244330606 0.4351881757862846

0.5650105307170965 0.6822087048389735

0.7841637256141400 0.8685966695651883

0.9336186965721600 0.9777691371910347

1.000000000000000

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